A general initial condition of inflationary cosmology on trans-Planckian physics

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Abstract

We consider a more general initial condition satisfying the minimal uncertainty relationship. We calculate the power spectrum of a simple model in inflationary cosmology. The results depend on perturbations generated below a fundamental scale, e.g. the Planck scale.

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In recent years it has been realized that tiny quantum fluctuations can be magnified by inflation. So we can study high energy physics through the cosmological observation. The results from WMAP will give us more precise cosmological parameters and the information which we can't get at the accelerators. At the very high energy scale, general relativity will break down and quantum gravity effects will appear. So we can get some information of trans-Planckian physic through cosmology.

The trans-Planckian problems in inflation was first raised explicitly in[1]. In general, there are three ways to tackle this problem. One is to modify the dispersion relations[2]-[3]. In this model the normal linear dispersion relation is replaced by new dispersion relations which differ from the linear one on length scales smaller than the Planck length.

Another line of approach to the trans-Planckian problems is to modify the equation of motion[4]-[10]. Inspired from string theory, space-time is noncommutative. So the space-space noncommutativity and the space-time noncommutativity was studied to get the modified power spectrum of cosmological fluctuations.

In yet another approach to the trans-Planckian issue, Danielsson[11]suggested the modified initial condition. In this model, the modes representing cosmological fluctuations are generated mode by mode at the time when the physical wavelength of the mode equals to the Planck length, or more generally, when the energy scale equals to a new physics scale. This kind of vacuum states are called α vacua. This model was discussed in many papers[12]-[27], the linear order corrections in H/Λ of power spectrum is got, where H is the Hubble constant.

In this paper, we assume that some fluctuations are already generated below the Planck scale and it is a state satisfying the minimum uncertainty relationship as the vacuum. This kind of quantum fluctuations can also be magnified by inflation. So the power spectrum of the cosmological fluctuations will be modified by this choice of initial condition.

Let us consider a simple case in de-Sitter space-time, the metric of space-time is

$$ds^2 = -a(\eta)^2 \left(d\eta^2 - dx^2\right) \tag{1}$$

where the η is the conformal time and $a(\eta) = -\frac{1}{\eta H}$. Consider a scalar field in this background, the action is

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\partial_\mu\phi\partial^\mu\phi \tag{2}$$

rescale the scalar field with $\mu = a\phi$, we get the action in $\mathbf{k}$ space

$$S = -\frac{1}{2}\int d\eta \int d^3\mathbf{k}\Big[\mu'_{\mathbf{k}}\mu^{\dagger\prime}_{\mathbf{k}} + \frac{a'^2}{a^2}\mu_{\mathbf{k}}\mu^{\dagger}_{\mathbf{k}} - \frac{a'}{a}(\mu'_{\mathbf{k}}\mu^{\dagger}_{\mathbf{k}} + \mu_{\mathbf{k}}\mu^{\dagger\prime}_{\mathbf{k}}) - k^2\mu_{\mathbf{k}}\mu^{\dagger}_{\mathbf{k}}\Big] \tag{3}$$

and the Hamitonian

$$H = \frac{1}{2}\int d^3k\Big[\pi'_{\mathbf{k}}\pi^{\dagger\prime}_{\mathbf{k}} + \frac{a'}{a}(\mu_{\mathbf{k}}\pi^{\dagger}_{\mathbf{k}} + \mu^{\dagger}_{\mathbf{k}}\pi_{\mathbf{k}}) + k^2\mu_{\mathbf{k}}\mu^{\dagger}_{\mathbf{k}}\Big] \tag{4}$$

where

$$\pi_{\mathbf{k}} = \mu'_{\mathbf{k}} - \frac{a'}{a}\mu_{\mathbf{k}}. \tag{5}$$

using the time dependent oscillators, we can write

$$\begin{aligned}
\mu_{\mathbf{k}}(\eta) &= \frac{1}{\sqrt{2k}}\left(a_{\mathbf{k}}(\eta) + a^{\dagger}_{-\mathbf{k}}(\eta)\right) \\
\pi_{\mathbf{k}}(\eta) &= -i\sqrt{\frac{k}{2}}\left(a_{\mathbf{k}}(\eta) - a^{\dagger}_{-\mathbf{k}}(\eta)\right)
\end{aligned} \tag{6}$$

The Hamitonian reads

$$H = \int d^3\mathbf{k}\Big[k a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} + \frac{i}{2}\frac{a'}{a}\Big(a^{\dagger}_{-\mathbf{k}} a^{\dagger}_{\mathbf{k}} - a_{-\mathbf{k}} a_{\mathbf{k}}\Big)\Big] \tag{7}$$

The creation and annihilation operators satisfy the Heinsenberg equations:

$$\begin{aligned}
i\frac{d}{d\eta}a_{\mathbf{k}}(\eta) &= [a_{\mathbf{k}}, H] \\
i\frac{d}{d\eta}a_{\mathbf{k}}(\eta)^{\dagger} &= [a^{\dagger}_{\mathbf{k}}, H]
\end{aligned} \tag{8}$$

The time dependence of the oscillators can be written

$$\begin{aligned}
a_{\mathbf{k}}(\eta) &= u_k(\eta)\, a_{\mathbf{k}}(\eta_0) + v_k(\eta)\, a^{\dagger}_{-\mathbf{k}}(\eta_0) \\
a^{\dagger}_{-\mathbf{k}}(\eta) &= u^*_k(\eta)\, a^{\dagger}_{-\mathbf{k}}(\eta_0) + v^*_k(\eta)\, a_{\mathbf{k}}(\eta_0)
\end{aligned} \tag{9}$$

where η_0 is some fixed initial time, e.g. the time associated with the Planck energy. Plugging this back into the Heinsenberg equations, we can get the solutions

$$\begin{aligned}
u_k &= \frac{1}{2}\left(A_k e^{-ik\eta}\left(2 - \frac{i}{k\eta}\right) + B_k e^{ik\eta}\frac{i}{k\eta}\right) \\
v^*_k &= \frac{1}{2}\left(B_k e^{ik\eta}\left(2 + \frac{i}{k\eta}\right) - A_k e^{-ik\eta}\frac{i}{k\eta}\right).
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
A_k &= (1 + \frac{i}{2k\eta_0})e^{ik\eta_0} \\
B_k &= \frac{i}{2k\eta_0}e^{-ik\eta_0}
\end{aligned} \tag{11}$$

Now, let us choose the initial condition. In[11], the choice is that the initial state is vacuum when $\eta = \eta_0$. This choice means that there are no fluctuations at all until the Planck scale. However, the physics is complex at Planck scale in string theory as we know. So the initial may be a more complex state but vacuum. If we assume

that some quantum fluctuations are generated below the Planck scale and keep the minimum quantum fluctuations before inflation:

$$\Delta\mu_k \Delta\pi_k = \frac{1}{2} \tag{12}$$

where

$$\begin{aligned} \Delta\mu_k^2 &= <\mu_{\mathbf{k}}^\dagger \mu_{\mathbf{k}}> - <\mu_{\mathbf{k}}^\dagger><\mu_{\mathbf{k}}> \\ \Delta\pi_k^2 &= <\pi_{\mathbf{k}}^\dagger \pi_{\mathbf{k}}> - <\pi_{\mathbf{k}}^\dagger><\pi_{\mathbf{k}}> \end{aligned} \tag{13}$$

Except for vacuum, the only choice is coherent state.

$$a_{\mathbf{k}}(\eta_0)\left|\alpha_{\mathbf{k}}, \eta_0\right\rangle = \alpha_{\mathbf{k}}\left|\alpha_{\mathbf{k}}, \eta_0\right\rangle \tag{14}$$

where

$$\alpha_{\mathbf{k}} = \sqrt{N_k} e^{i\theta_{\mathbf{k}}} \tag{15}$$

N_k is the particle number representing the amplitude of the fluctuations and $\theta_{\mathbf{k}}$ is the phase factor representing the phase of the fluctuations. The Power spectrum is

$$\begin{aligned} P_\phi &= \frac{k^3}{2\pi^2 a^2}\left\langle \mu_{\mathbf{k}}^\dagger \mu_{\mathbf{k}} \right\rangle \\ &= \frac{k^2}{4\pi^2 a^2}\left\langle \left((u_k + v_k^*)\, a_{\mathbf{k}} + (u_k^* + v_k)\, a_{-\mathbf{k}}^\dagger \right)\left((u_k^* + v_k)\, a_{\mathbf{k}}^\dagger + (u_k + v_k^*)\, a_{-\mathbf{k}} \right)\right\rangle \end{aligned} \tag{16}$$

For $\eta \to 0$, $v_k^*(\eta) \to u_k(\eta)$, the spectrum reads

$$P_\phi = \frac{k^2}{\pi^2 a^2}\left|u_k\right|^2 \left[1 + N_k\left(2 + e^{i(2\theta(\eta)+\theta_{\mathbf{k}}+\theta_{-\mathbf{k}})} + e^{-i(2\theta(\eta)+\theta_{\mathbf{k}}+\theta_{-\mathbf{k}})}\right)\right] \tag{17}$$

where $\theta(\eta)$ is the phase of u_k, $u_k = |u_k|\, e^{i\theta(\eta)}$. In bracket, the first term represents the fluctuations generated after η_0, the second term proportional to N_k represents the earlier quantum fluctuations magnified by inflation. It is proportional to $<\mu_{\mathbf{k}}^\dagger><\mu_{\mathbf{k}}>$. If the phase factors of the initial coherent state satisfy $\theta_{\mathbf{k}} + \theta_{-\mathbf{k}} = -2\theta(\eta) + \pi$, it implies $<\mu_k>= 0$ at late times. We can see that the correction proportional to N_k totally vanishes. So we can't see the fluctuations generated below Planck scale. But this condition seems to be not reasonable. Why we should choose the initial phase factor at $\eta = \eta_0$ satisfying $<\mu_{\mathbf{k}}>$ at late times? Now let us consider the Hamitonian of a fixed $\mathbf{k}$ mode. At $\eta = \eta_0$, the energy is

$$E|_{\eta=\eta_0} = \langle H|_{\eta=\eta_0}\rangle = kN_k\left[1 - \frac{1}{k\eta_0}\sin(\theta_{\mathbf{k}} + \theta_{-\mathbf{k}})\right] \tag{18}$$

The same as [11], where $k\eta_0 = -\frac{\Lambda}{H}$.

$$E|_{\eta=\eta_0} = kN_k\left[1 + \frac{H}{\Lambda}\sin(\theta_{\mathbf{k}} + \theta_{-\mathbf{k}})\right] \tag{19}$$

So the more reasonable initial condition is $\theta_{\mathbf{k}}+\theta_{-\mathbf{k}}=\frac{3}{2}\pi$, which minimize the energy at $\eta=\eta_0$. This condition also set $<\mu_{\mathbf{k}}>=0$ at $\eta=\eta_0$. Plugging this initial condition into (17), we can get

$$P_\phi=\frac{k^2}{\pi^2 a^2}\left|u_k\right|^2\left[1+2N_k\left(1+\sin 2\theta(\eta)\right)\right] \tag{20}$$

Expanding the result in first order of $\frac{H}{\Lambda}$, we can get

$$P_\phi=\left(\frac{H}{2\pi}\right)^2\left[1-\frac{H}{\Lambda}\sin(\frac{2\Lambda}{H})+2N_k(1+\sin\frac{2\Lambda}{H}-\frac{H}{\Lambda}\sin\frac{2\Lambda}{H}+\frac{H}{\Lambda}\cos\frac{2\Lambda}{H}-\frac{H}{\Lambda})\right] \tag{21}$$

Some comments on this results are in order. First, the power spectrum depends on the phase factor of the initial state. As we know, most of the physical observable quantities are independent of the phase factor in quantum mechanics. But we can see that the phase factor will affect the power spectrum. Different choices of phase factor result in different corrections.

Second, the first order correction in [11] proportions to $\sin\frac{2\Lambda}{H}$. Because $\frac{\Lambda}{H}$ is a very large number, the different choices of Λ result in a big oscillations in correction terms. It seems unphysical. In our results, we get a correction term proportional to N_k, it is independent of the choice of Λ. Interestingly, If we adjust the $N_k=\frac{H}{2\Lambda}$, we can get the linear correction of $\frac{H}{\Lambda}$.

$$P_\phi=\left(\frac{H}{2\pi}\right)^2\left(1+\frac{H}{\Lambda}\right) \tag{22}$$

But we don't know how to determine N_k which represent the magnitude of the fluctuations below Planck scale. It may rely on the details of high-energy physics.

Now, a short conclusion can be made. In this paper, we assume that some quantum fluctuations are generated below the new physics scale. Because we know little about the high-energy physics, we assume that all of high-energy physics information is imprinted effectively in the initial state. To maintain the minimal uncertainty relationship, we set the coherent state as the initial state. This kind of quantum fluctuations also can be magnified by inflation. The result depend on the eigenvalue of the coherent state. If the fluctuations generated below Planck scale are big enough, i.e. N_k is big enough, the effect of trans-Planckian physics are possibly observed by more precise cosmological observation. It is possibly a window through which we can see the quantum gravity information.

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